FEM Analysis of Stress Distribution in Force-Balanced Coils

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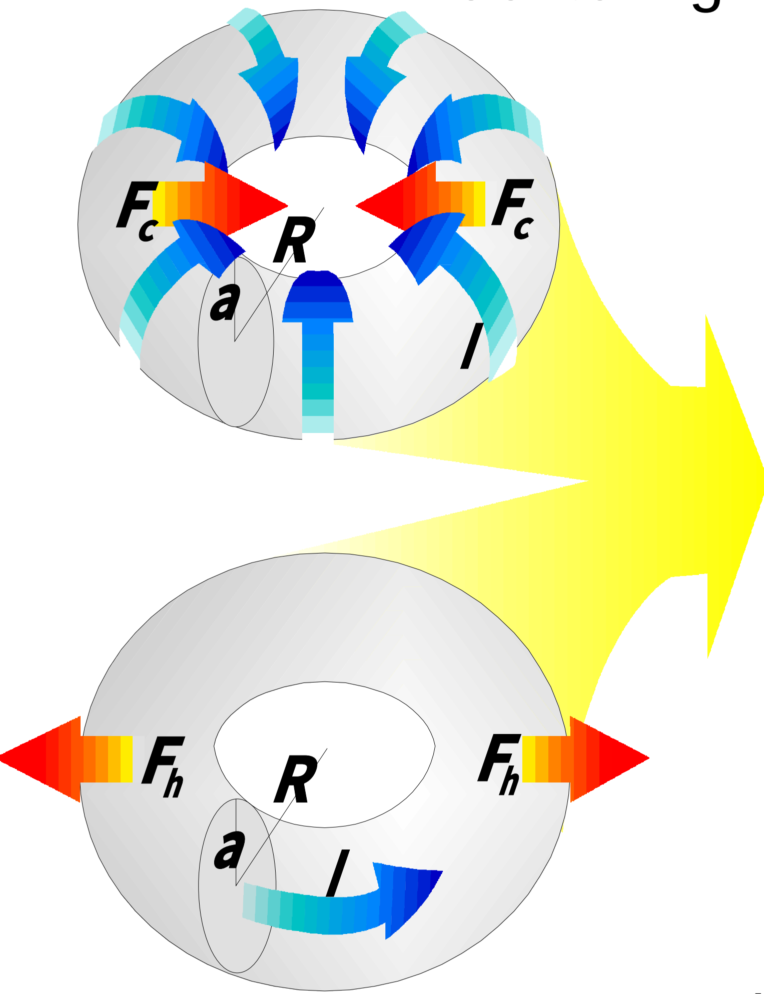
Introduction

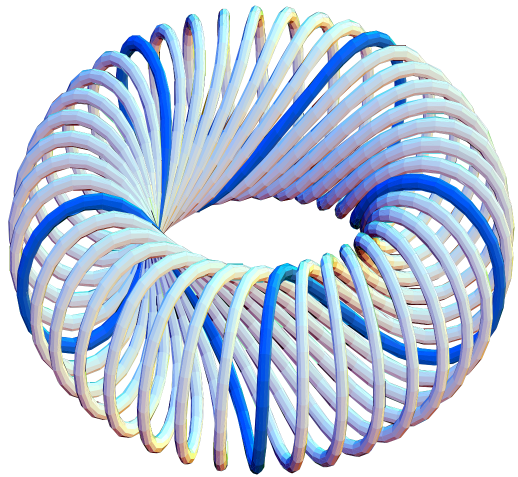
- The **virial theorem** is the relation between the kinetic and the potential energies. The theorem, which is derived only form the equilibrium, shows that the tension is required to hold the magnetic energy.
- Using the **virial theorem,** we extended and generalized Force-Balanced Coil which is a helical type hybrid coil of the toroidal field (TF) coil and the solenoidal coil, and showed the condition to minimize the stress working in the coil (**virial-limit condition**).
- In this work, we constructed a small device to prove our concept, and obtained stress distribution experimentally. The results are compared with those of numerical calculations with a shell model.

Principle of Force-Balanced Coil

Centering Force by Poloidal Current



Hoop Force by Toroidal Current



Force-Balanced Coil

Centering force is much reduced,

but

stress distribution is not investigated.

Virial Theorem

$$\boldsymbol{j}\times\boldsymbol{B}+\nabla\cdot\mathsf{S}=0$$
$$\nabla\times\boldsymbol{B}=\mu_0\boldsymbol{j}$$
$$\nabla\cdot\boldsymbol{B}=0$$

$\boldsymbol{j}$: current density

$\boldsymbol{B}$: magnetic field

S: stress tensor

Equilibrium Eq.

$$\nabla\cdot(\mathsf{T}+\mathsf{S})=0$$
$$\mathsf{T}\equiv\frac{1}{\mu_0}\left(\boldsymbol{BB}-\frac{B^2}{2}\mathsf{I}\right)$$

T: Maxwell stress tensor

$$\int\mathrm{Tr}(\mathsf{T}+\mathsf{S})\,\mathrm{d}V=0$$
$$\int\sum_{i=1}^{3}\sigma_i\,\mathrm{d}V=\int\frac{B^2}{2\mu_0}\mathrm{d}V\equiv U_\mathrm{M}>0$$

σ_i : principal stress

$$\tilde{\sigma}\equiv\frac{V_\Omega}{U_\mathrm{M}}\sigma$$
$$\langle\sigma\rangle\equiv\frac{\int\sigma\,\mathrm{d}V}{V_\Omega}$$

$$\left\langle\sum_{i=1}^{3}\tilde{\sigma}_i\right\rangle=1$$

- Positive stress (tension) is required to hold the field.
- Uniform tension is favorable.
- Theoretical limit is determined.

$$\tilde{\sigma}_1=\tilde{\sigma}_2=\tilde{\sigma}_3=\frac{1}{3}$$

Application to Thin Toroidal Shell

- We consider the toroidal coil with so large aspect ratio that toroidal effect is negligible.
- The current distribution is adopted which makes toroidal surface correspond to both current and magnetic surfaces.
- When torus is axisymmetric, the direction of principal stresses are ϕ and θ.

TF Coil (Major radius: *R*, minor radius: *a*, thickness: Δρ)

	ϕ	θ	Sumation
Stress	$-\frac{\mu_0 a^2 I_\theta^2}{16\pi^2 R\Delta\rho}$	$\frac{\mu_0 a^2 I_\theta^2}{8\pi^2 R\Delta\rho}$	
Integral	$-\frac{\mu_0 a^2 I_\theta^2}{4R}$	$\frac{\mu_0 a^2 I_\theta^2}{2R}$	$\frac{\mu_0 a^2 I_\theta^2}{4R}$
Energy			$\frac{\mu_0 a^2 I_\theta^2}{4R}$

Application to Helical Coil

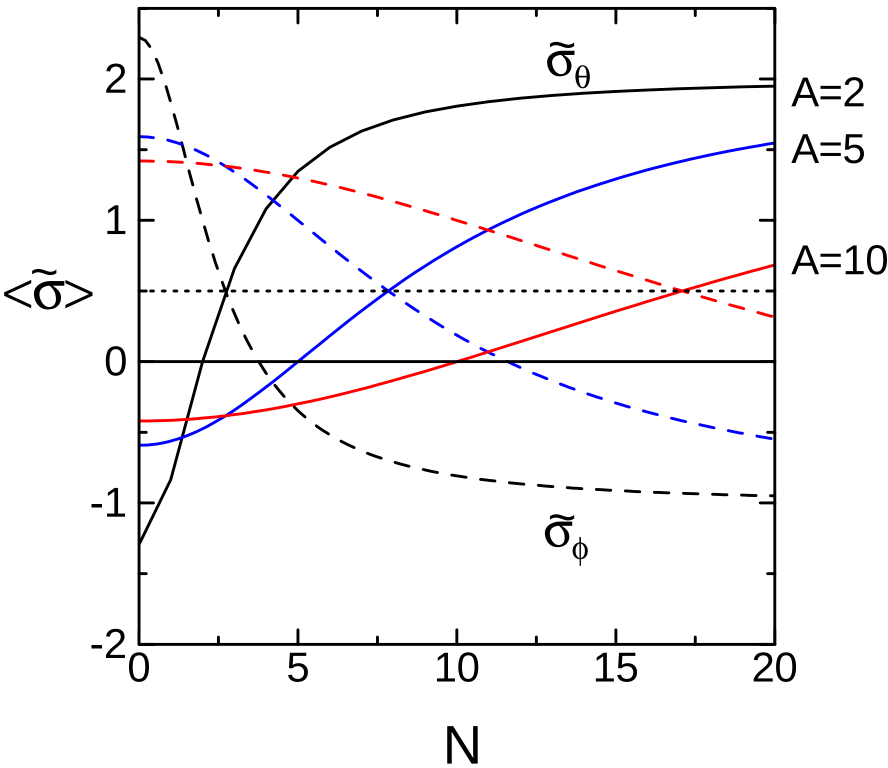
$$\left\langle \tilde{\sigma}_\theta \right\rangle = \frac{N^2 - A^2}{\dfrac{N^2}{2} + A^2 \log 8A - 2A^2}$$

$$\left\langle \tilde{\sigma}_\phi \right\rangle = \frac{A^2 \log 8A - A^2 - \dfrac{N^2}{2}}{\dfrac{N^2}{2} + A^2 \log 8A - 2A^2}$$

$$\left\langle \tilde{\sigma}_\theta \right\rangle + \left\langle \tilde{\sigma}_\phi \right\rangle = 1$$

$N \equiv \dfrac{I_\theta}{I_\phi}$: Pitch of Coil

A : Aspect Ratio

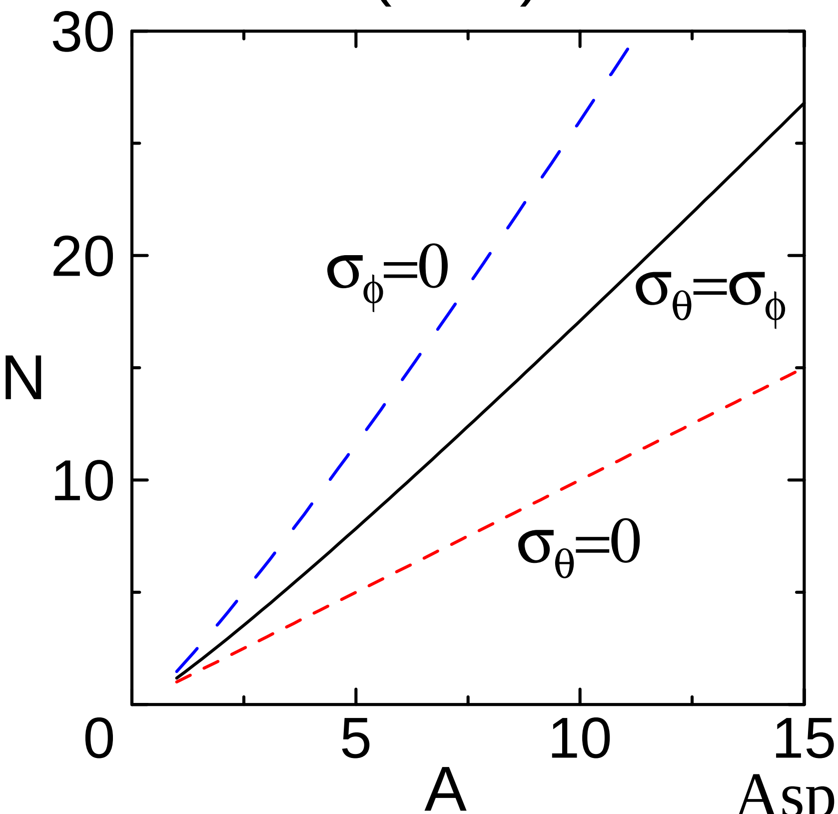


$\left\langle \tilde{\sigma}_\theta \right\rangle = \left\langle \tilde{\sigma}_\phi \right\rangle = \dfrac{1}{2}$ is optimal in energy.

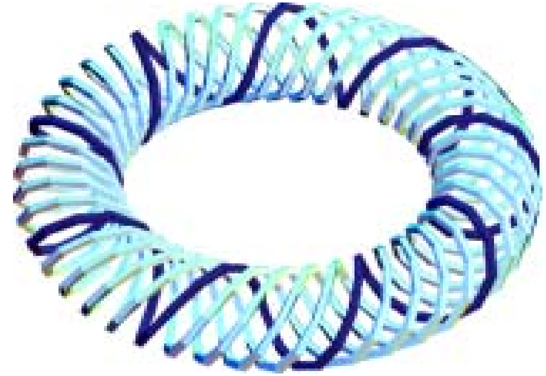
Virial-Limit Condition

Shape of Coils

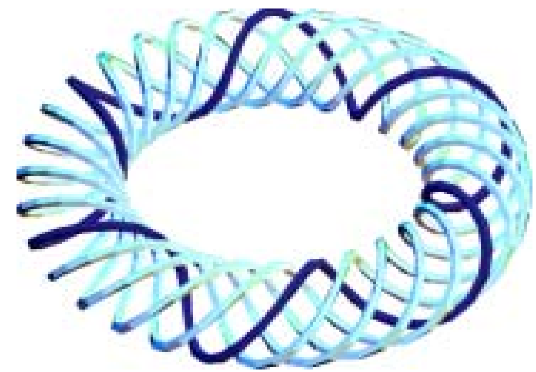
Relations of pitch number and aspect ratio of **Virial-Limit Coil** (**VLC**) etc.



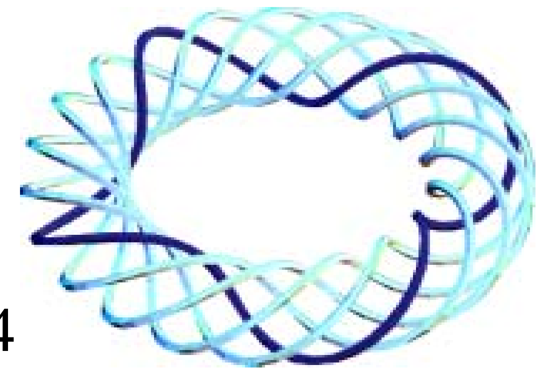
$<\sigma_\phi>=0$
N=9
FBC



$<\sigma_\phi>=<\sigma_\theta>$
N=6
VLC

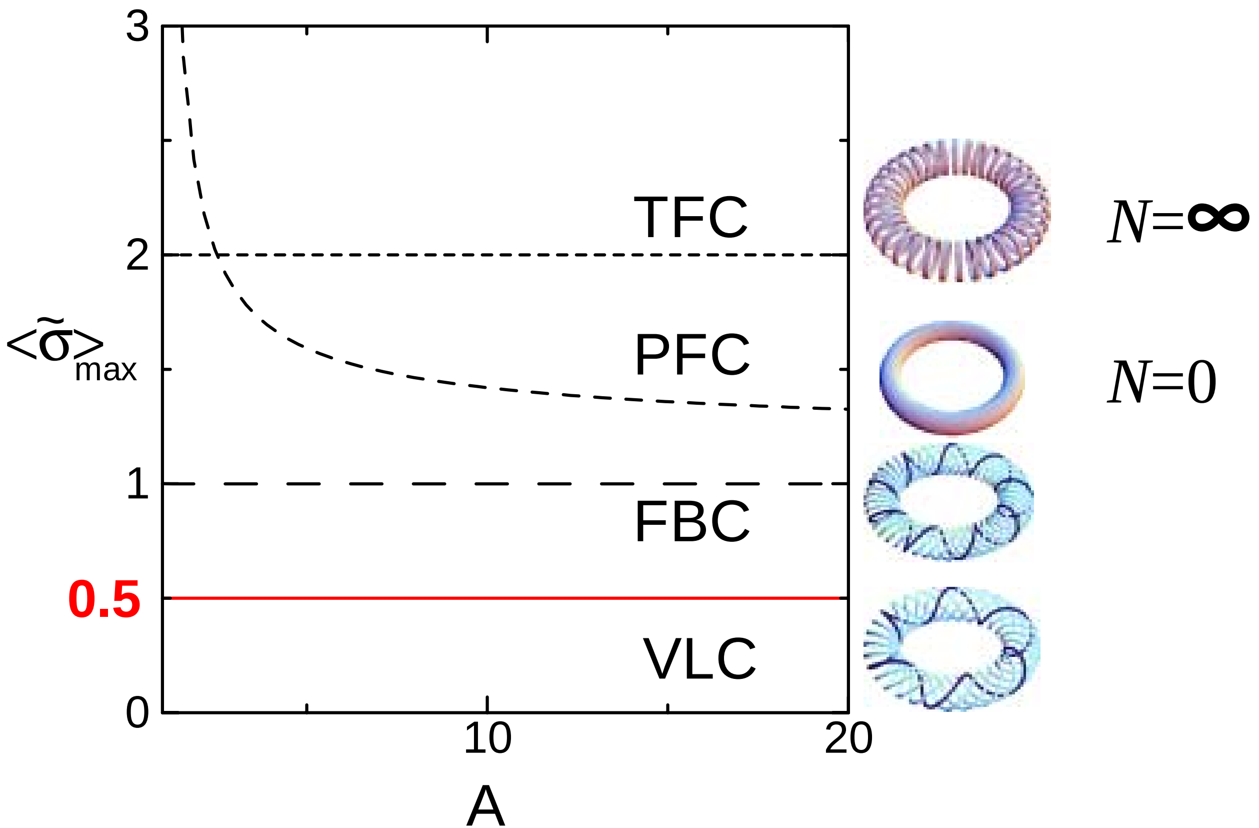


$<\sigma_\theta>=0$
N=4



Aspect Ratio A =4

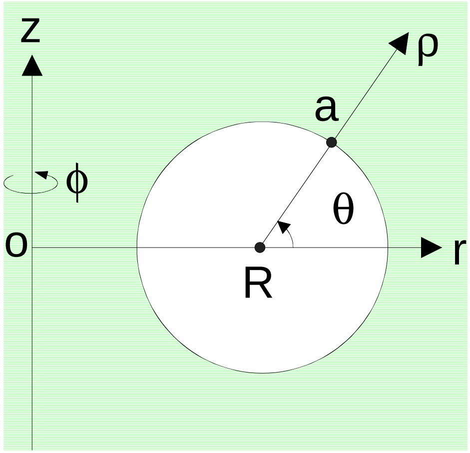
Comparison of Maximum Averaged Stress



- Neglecting the distribution of stress by the toroidal effect, the maximum stress is reduced to 25% compared with that of traditional TF coil.

Toroidal Effect

Equilibrium of Magnetic Pressure and Stress



$$u(r) \equiv a\int_R^r r' \, p(r')\mathrm{d}r'$$

$$T_\theta = \frac{u}{(r-R)r}$$

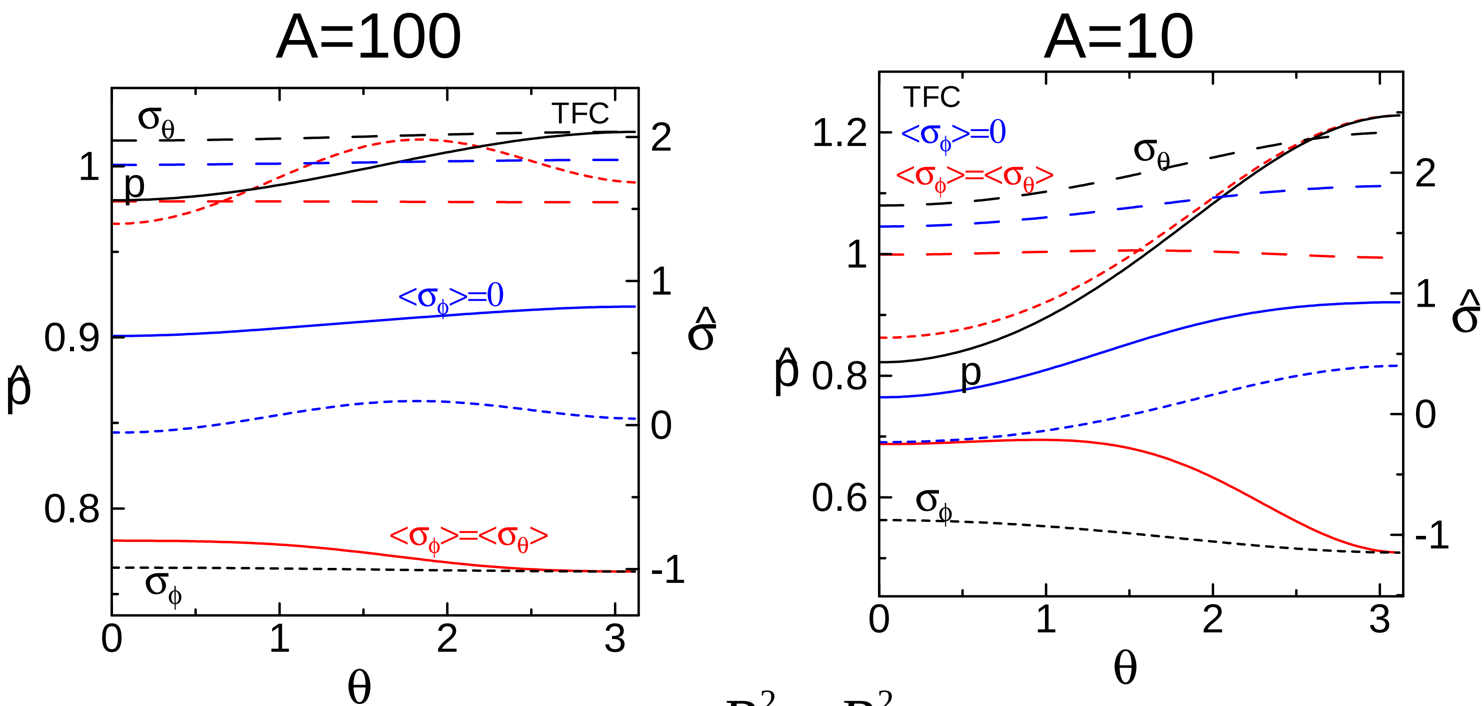
$$T_\phi = \frac{arp}{r-R} - \frac{u}{(r-R)^2}$$

$$p \equiv \frac{B_\phi^2 - B_\theta^2}{2\mu_0}$$

- Distribution of stress in the toroidal shell with circular cross section is derived analytically by use of magnetic pressure.

Distribution of Stress

(large aspect ratio)

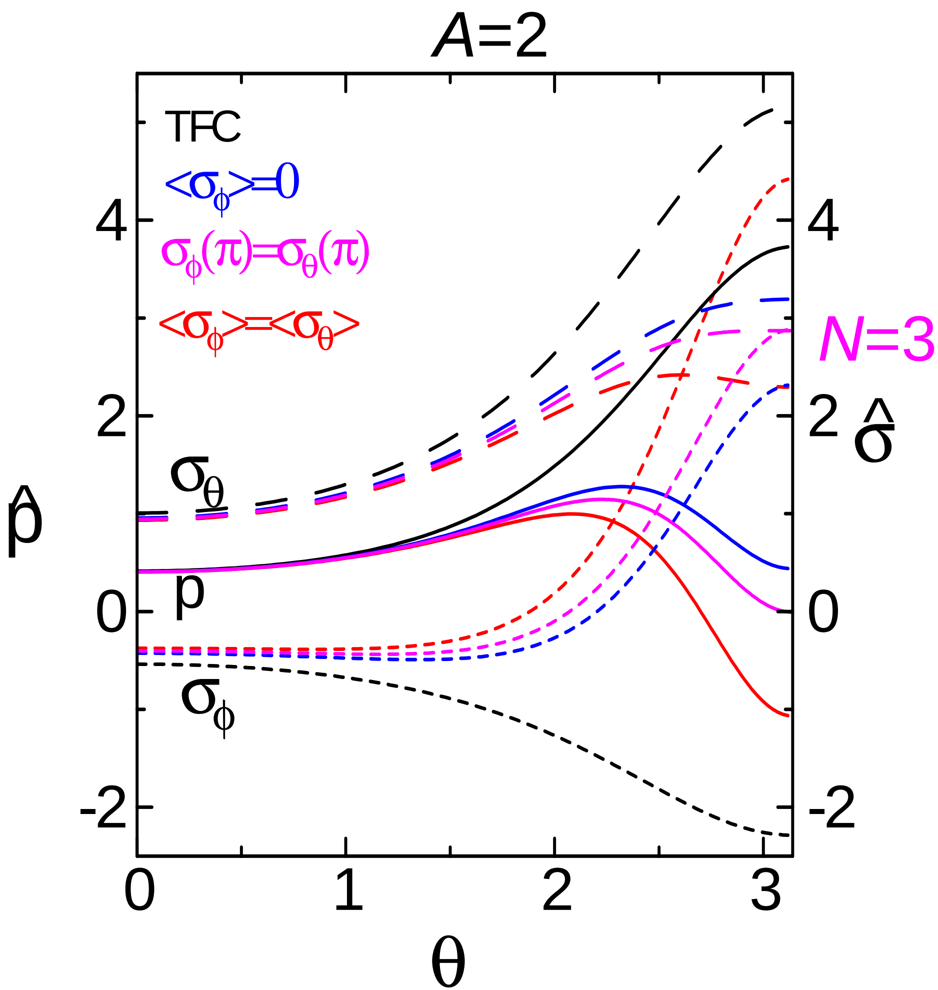


$$\hat{p} \equiv \frac{V_\mathrm{T}}{U_\mathrm{M}} \frac{B_\phi^2 - B_\theta^2}{2\mu_0}$$

- When A=100, distribution of stress is flat.
- There is no advantage of helical winding.

Distribution of Stress

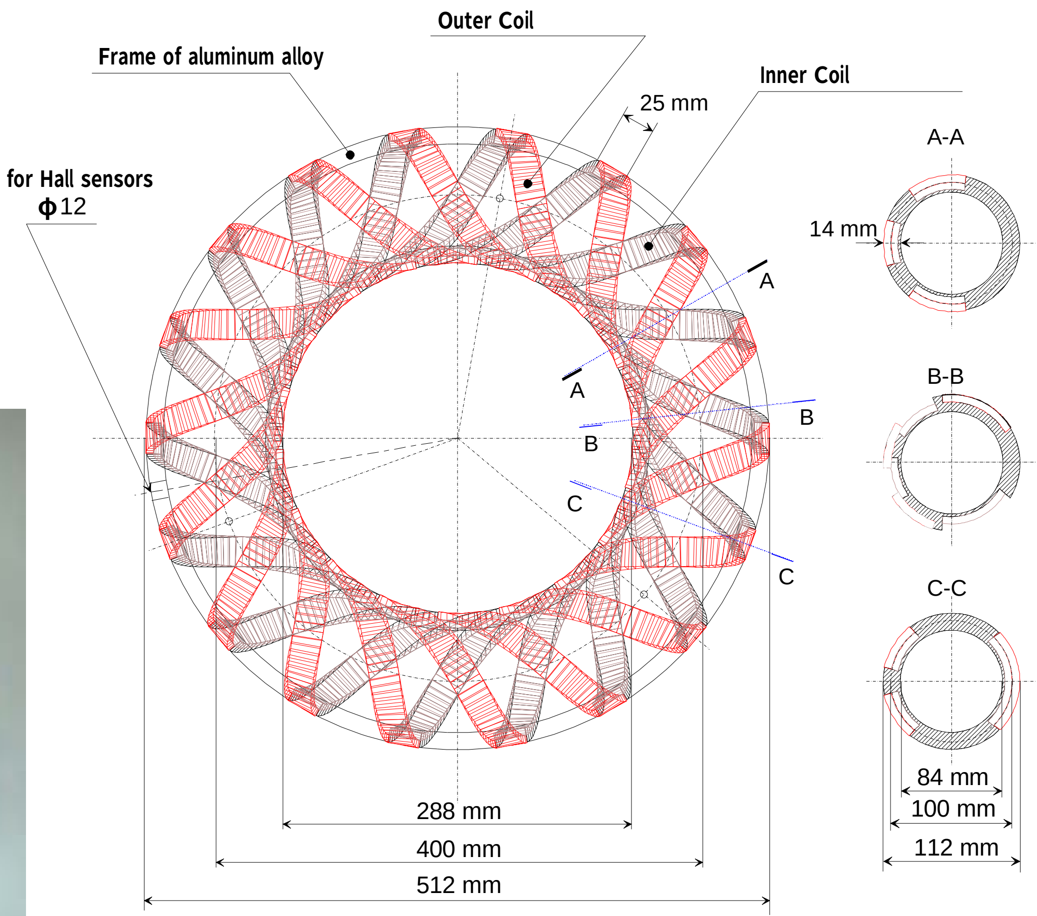
(low aspect ratio)



- When A<10, distribution of stress is important.
- Assumption of large aspect ratio is not held.
- Optimal distribution is achieved to minimize the stress at $\theta=\pi$.

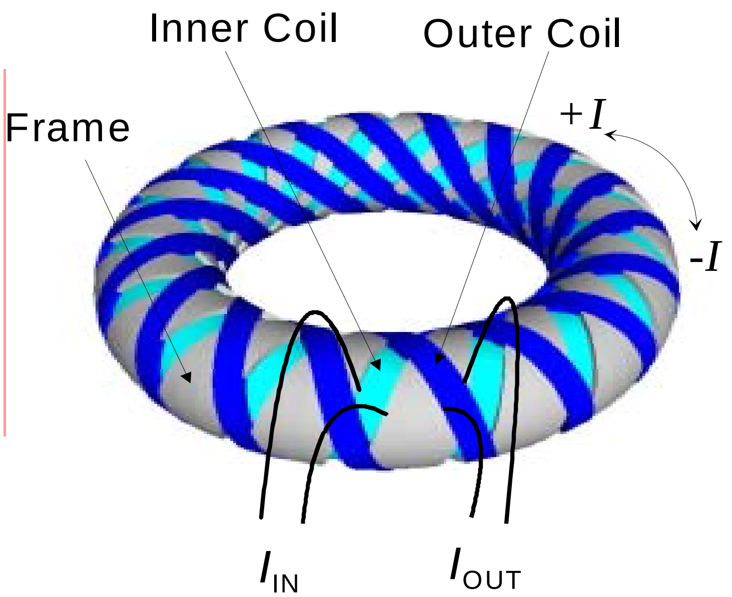
Frame of Device to demonstrate VLC

- It is made of Al with 2mm-14mm thickness so that the strain is 10^{-5} -10^{-4} when the maximum field is 1T.



Device to demonstrate Virial-Limit Condition

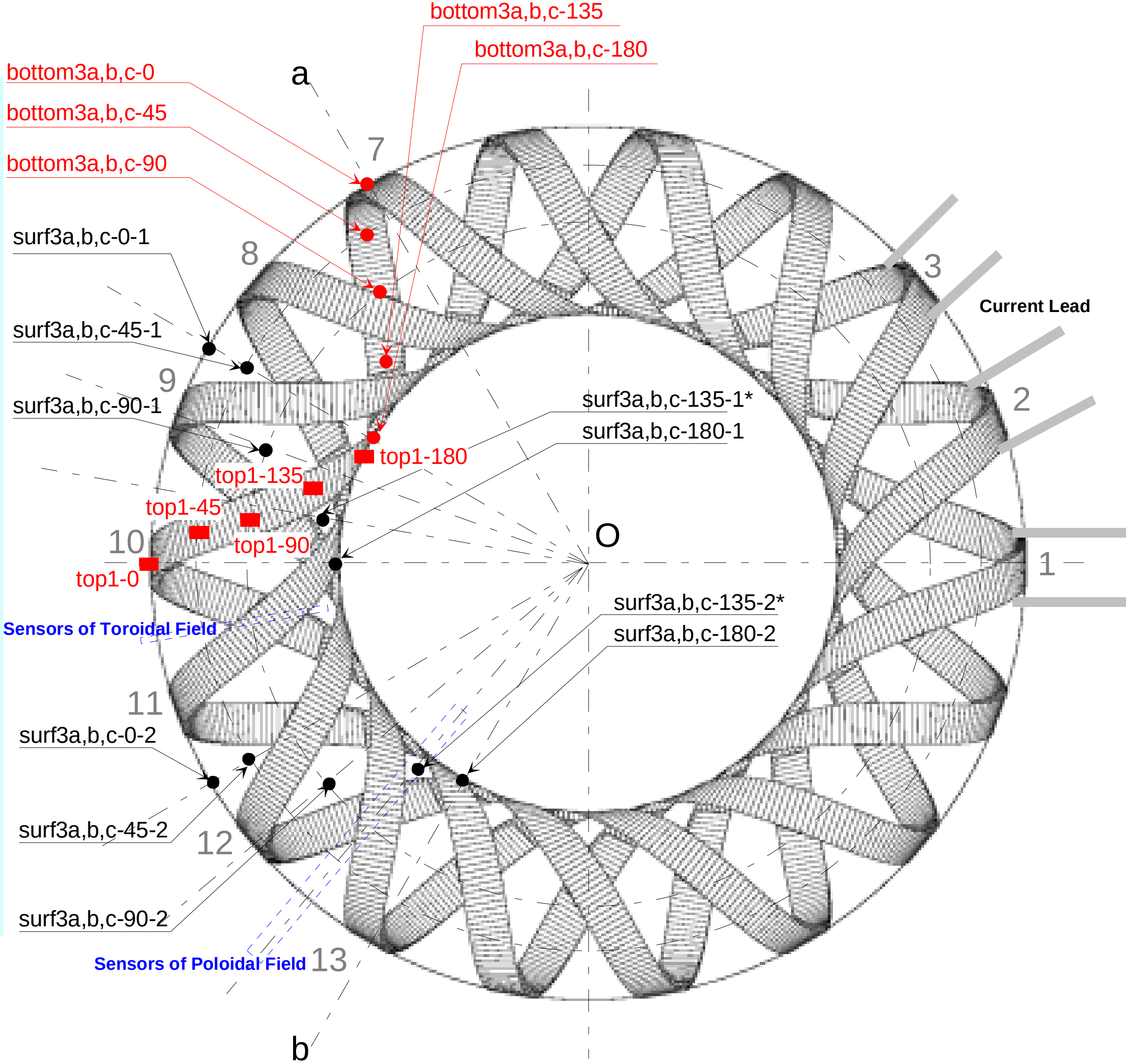
I_{out}/I_{in}	N	maximum stress	simulated coil
-1	0	$\tilde{\sigma}_\phi > 1$	PFC
-0.2	4	$\tilde{\sigma}_\phi = 1$	SBC
0	6	$\tilde{\sigma}_\theta = \tilde{\sigma}_\phi = 1/2$	VLC
0.2	9	$\tilde{\sigma}_\theta = 1$	FBC
1	∞	$\tilde{\sigma}_\theta = 2$	TFC



Coil of NbTi	
major/minor radii R, a	200 mm / 50 mm
pitch number N	6
the number of coils	3(inner) + 3(outer)
Frame of aluminum alloy	
Young's modulus E	70 GPa
Poisson's ratio ν	0.33
thickness $\Delta\rho$	14, 8, 2 mm

Positions of Strain Gauges

- In this work, data of triaxial strain gauges on the surface are used.
- surf3abc-0-1
- surf3abc-45-1
- surf3abc-90-1
- surf3abc-135-1
- surf3abc-180-1



Shell model with periodic symmetry

$$
\begin{aligned}
rT_\theta + (r-R)T_\phi &= arp(r,\phi), \\
\frac{\mathrm{d}r}{\mathrm{d}\theta}\left\{\frac{\partial}{\partial r}(rT_\theta) - T_\phi\right\} + a\frac{\partial S}{\partial \phi} &= 0, \\
\frac{\mathrm{d}r}{\mathrm{d}\theta}\left\{\frac{\partial}{\partial r}(rS) + S\right\} + a\frac{\partial T_\phi}{\partial \phi} &= 0, \\
R + a\cos\theta &= r,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}(rT_\theta) + aT_\phi \sin\theta + inaS &= 0, \\
\frac{\mathrm{d}}{\mathrm{d}\theta}(rS) - aS\sin\theta + inaT_\phi &= 0.
\end{aligned}
$$

Equilibirum Equation

where

$$T_\theta \equiv \sigma_{\theta\theta}\Delta\rho, \quad T_\phi \equiv \sigma_{\phi\phi}\Delta\rho, \quad S \equiv \sigma_{\phi\theta}\Delta\rho = \sigma_{\theta\phi}\Delta\rho$$

Expanding by $e^{in\phi}$

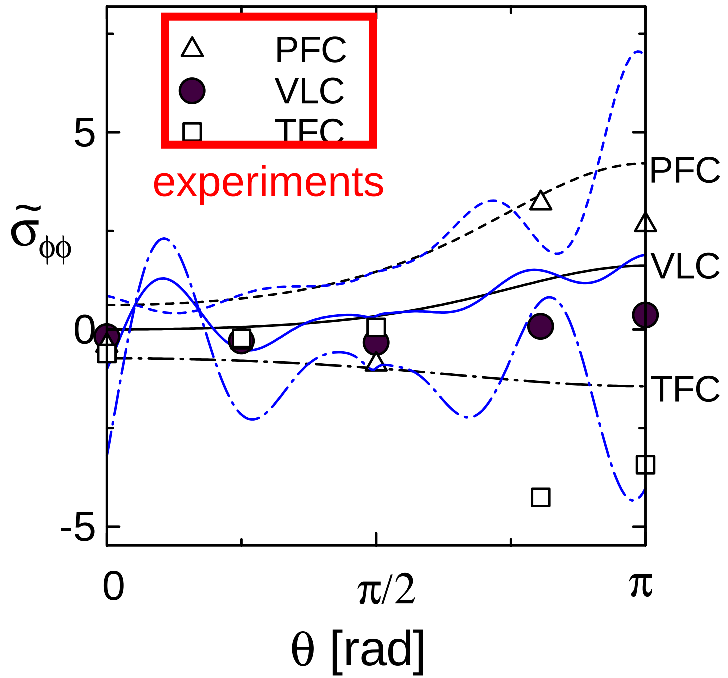
$$p(\theta,\phi) = p_0(\theta)\,(1 + \alpha\sin m\theta)\cos n\phi,$$

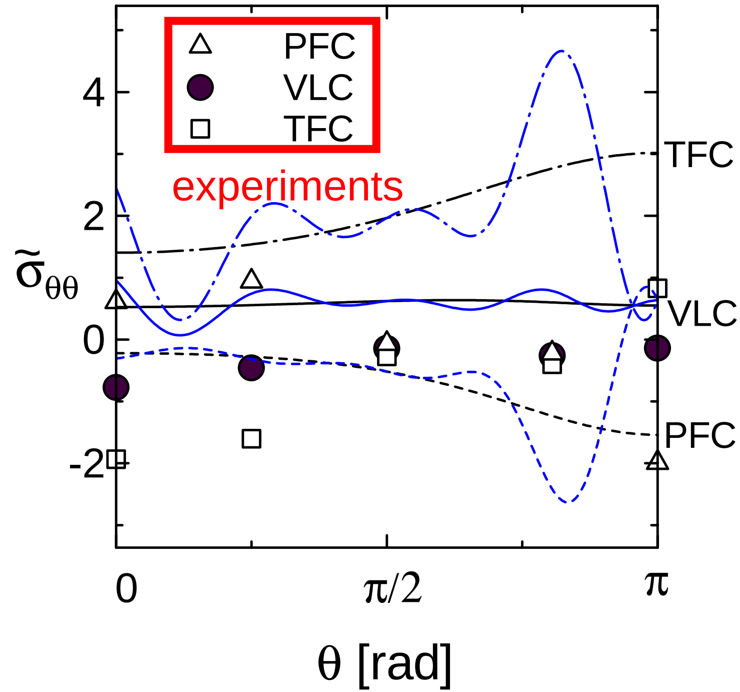
α: amplitude of mode

m,*n*: poloidal /todoidal mode numbers

Distribution of Stress

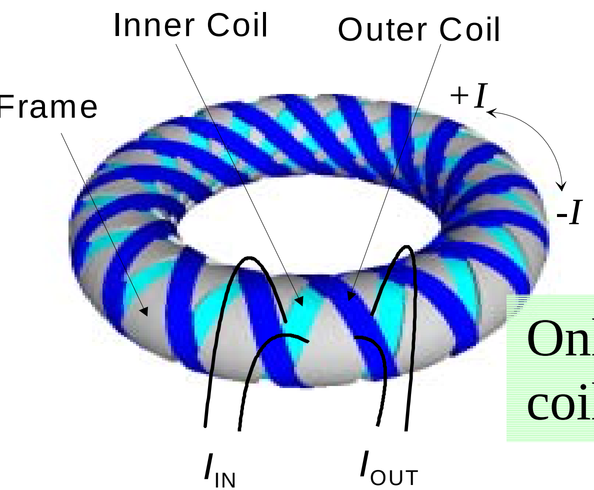
Lines: shell model with *A*=4, α=0.1, *m*=6, *n*=18





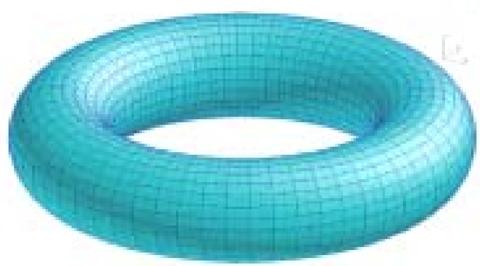
- Comparing the results of the experiments and the numerical calculations, a qualitative agreement of stress distribution between the calculation and the experiment is obtained in the toroidal direction, while discrepancies of stress in the poloidal direction are not negligible.

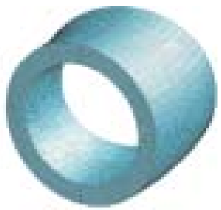
Model in FEM Analysis

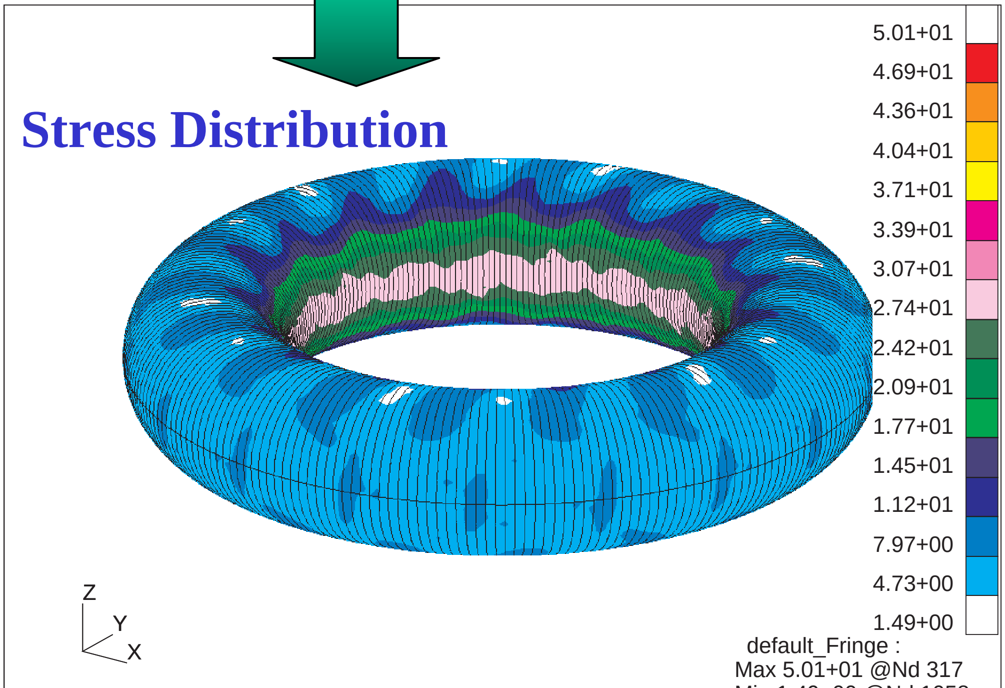


Only the **outer** coil is excited.

Electromagnetic force is applied to conductor region.

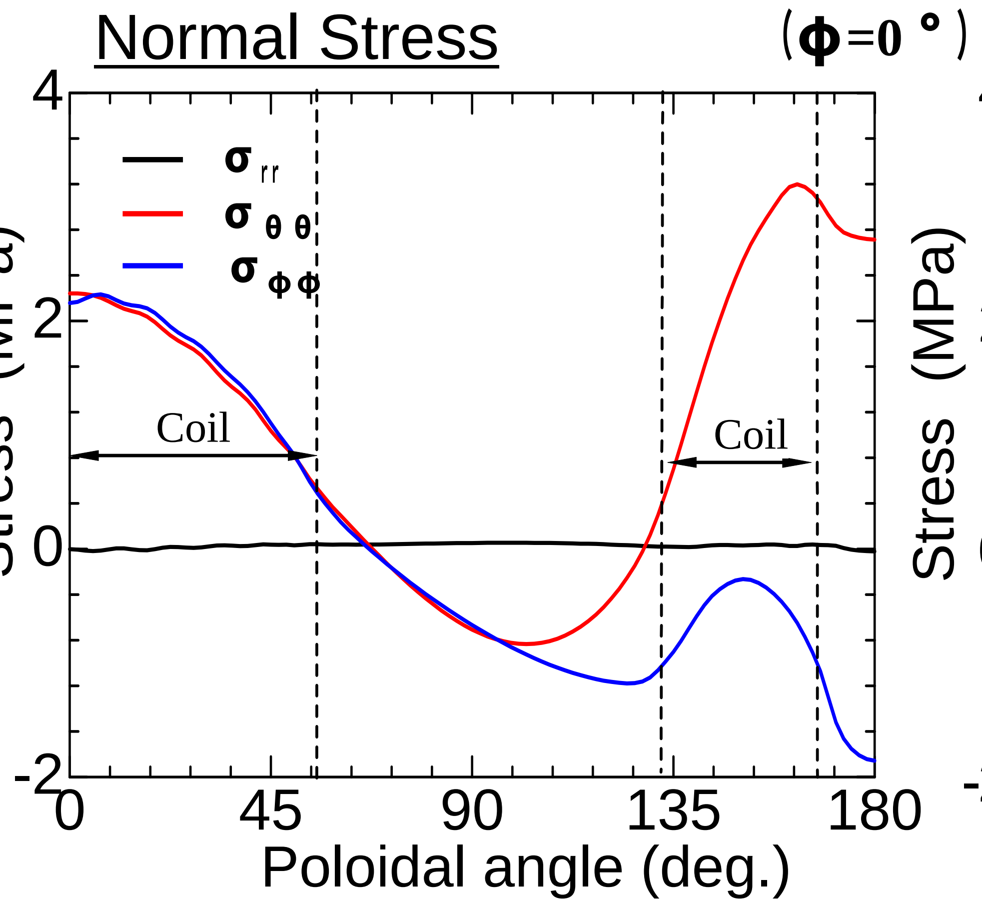


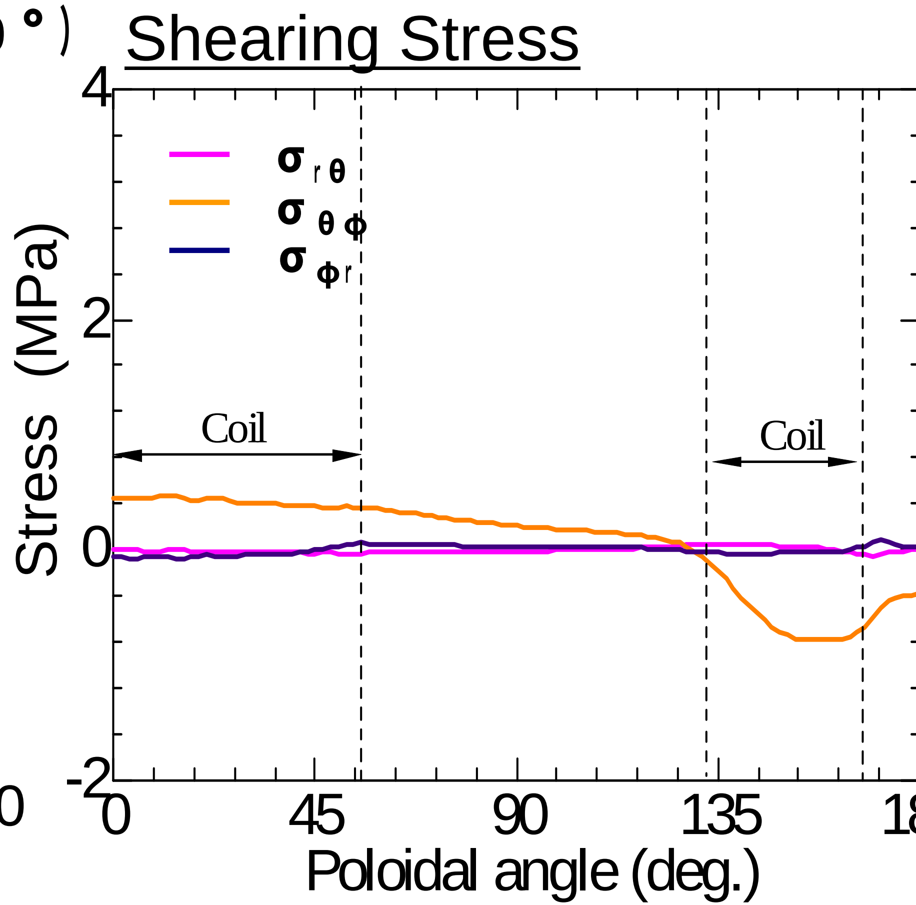




- Coils and their winding form are replaced with a **monolithic** shell.

Stress distribution on the coil-shell system

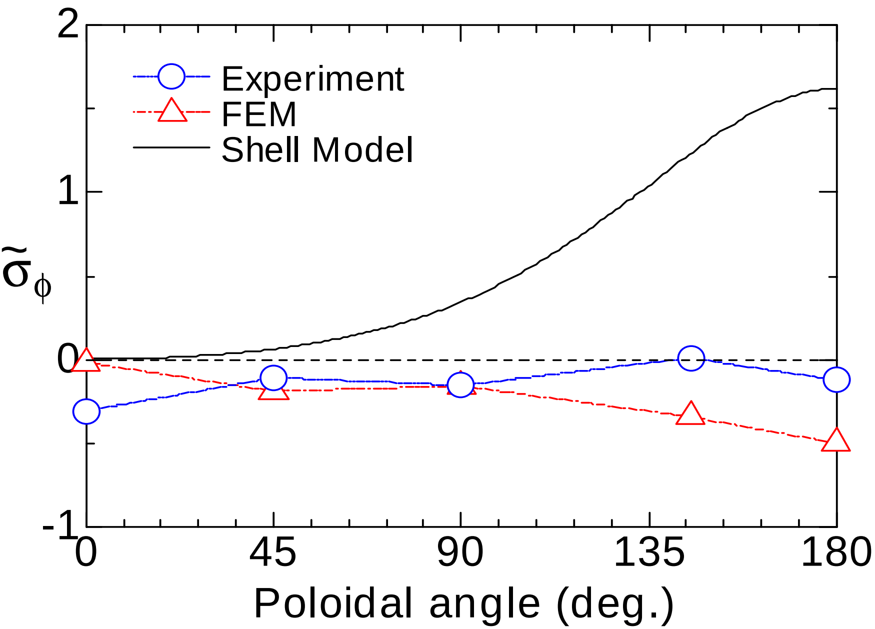




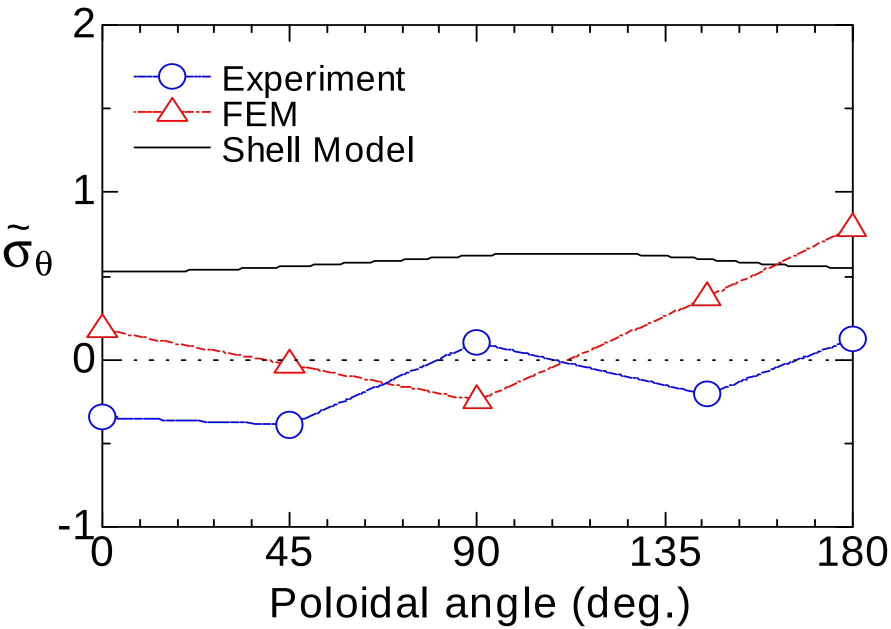
Normal stresses ($\sigma_{\theta\theta}$, $\sigma_{\theta\phi}$, $\sigma_{\phi\phi}$) are dominant compared with shearing stresses.

Comparison between FEM and Experiment

Toroidal direction

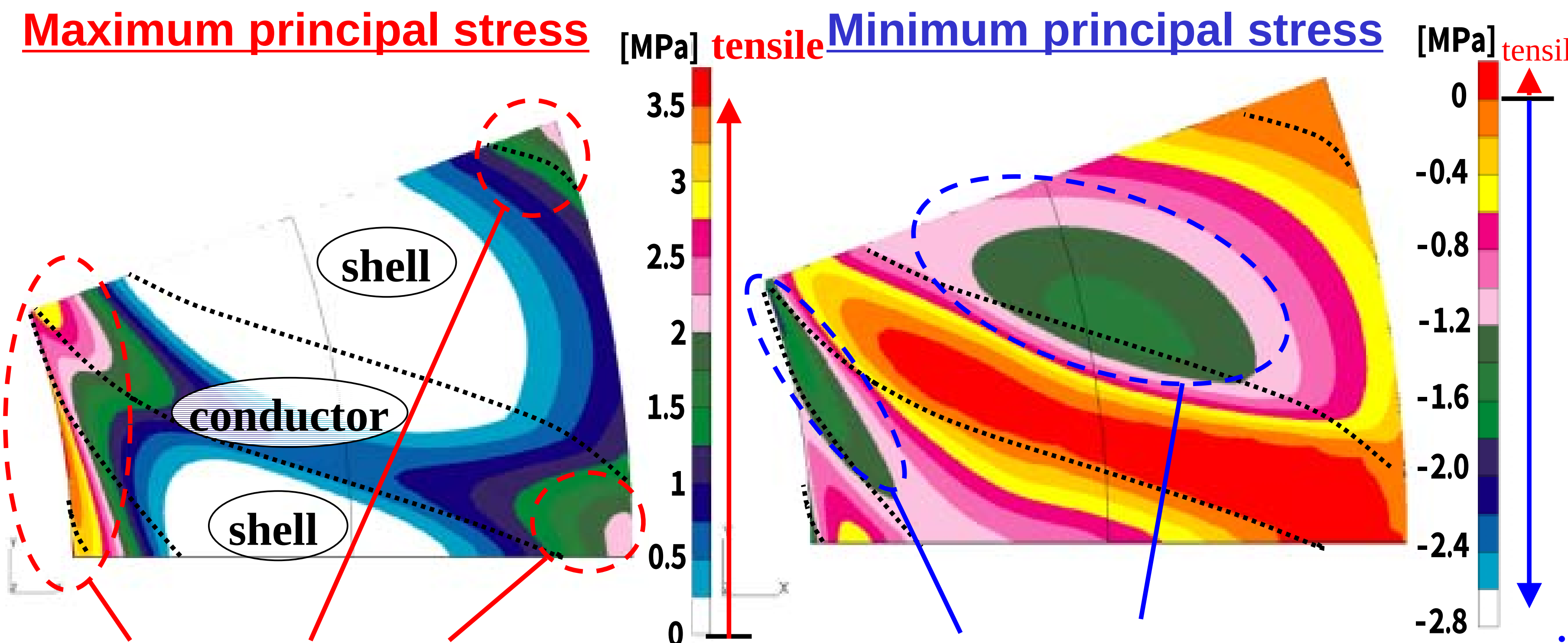


Poloidal direction



FEM analysis can reconstruct experimental results.

Principal stress on upper surface



Tensile stress is large

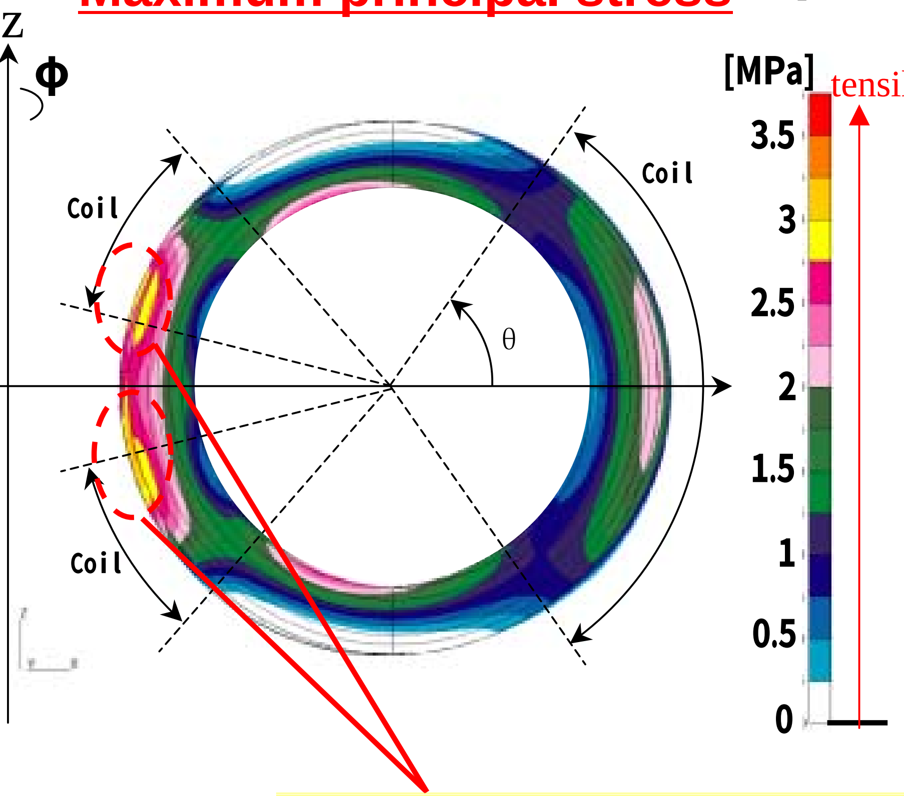
Compressive stress is large.

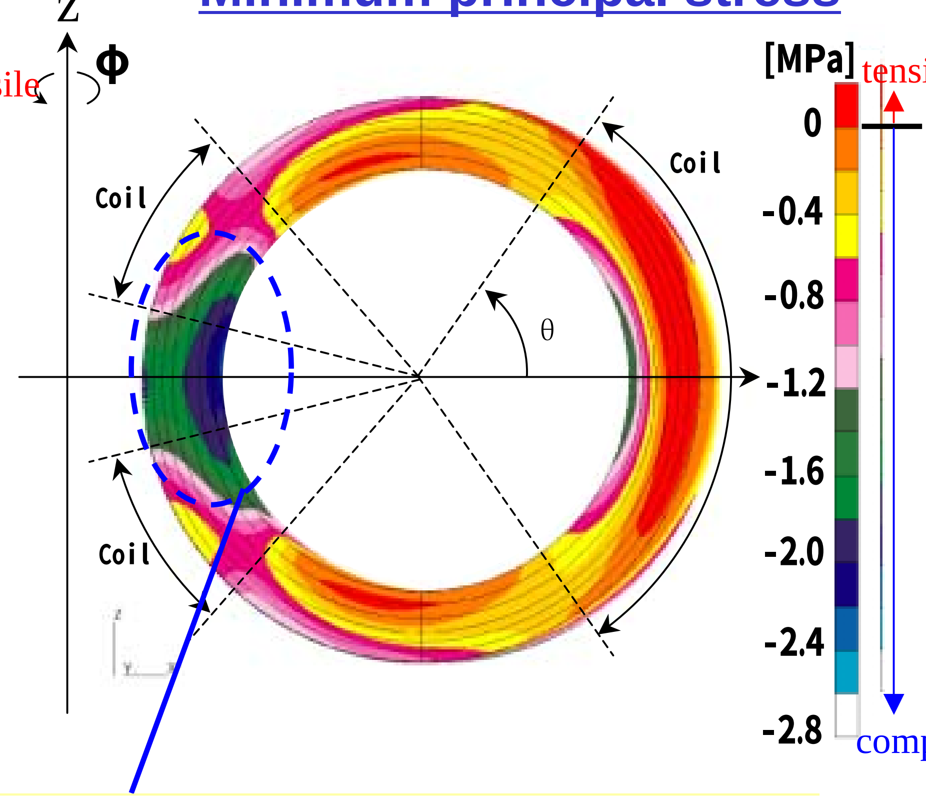
Tensile stress is large around the conductor, while compressive stress is large on the shell.

- Compressive stress is large on the shell where strain is measured.
- Stress distribution strongly depends on the location of conductors.

Principal stress on poloidal cross section

Maximum principal stress (ϕ = 0°) **Minimum principal stress**





Tensile stress is large in the conductor, while compressive stress is large in the form. Both stresses are concentrated around $\phi=\pi$ radian where magnetic field takes maximum.

Summary I

- In order to verify the concept of the optimal coil to store magnetic energy based on the virial theorem, the device composed of a toroidal winding-form and two sets of helical coils wound on two layers was constructed.
- The experiments with the device show that our optimal coil (VLC) achieves the minimum and flat stress distribution, which is also obtained by the numerical calculations with the shell model.
- In order to complete the VLC concept, stress distribution is evaluated by FEM with a monolithic model, and good agreement with the experiment is obtained.

Summary II

FEM analysis shows that;

- Compressive stress is produced mainly on the surface of the winding form where strains are measured, while tensile stress is produced mainly on the surface of the coil conductors.
- Stress distributions are quite different on inner and outer surfaces. Compressive stress is mainly produced in the shell region, and large compressive stress is produced around inner region where magnetic field takes the maximum value.
- A dense winding of coil conductors is required to reduce the compressive stress which is unfavorable for SMES.